2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

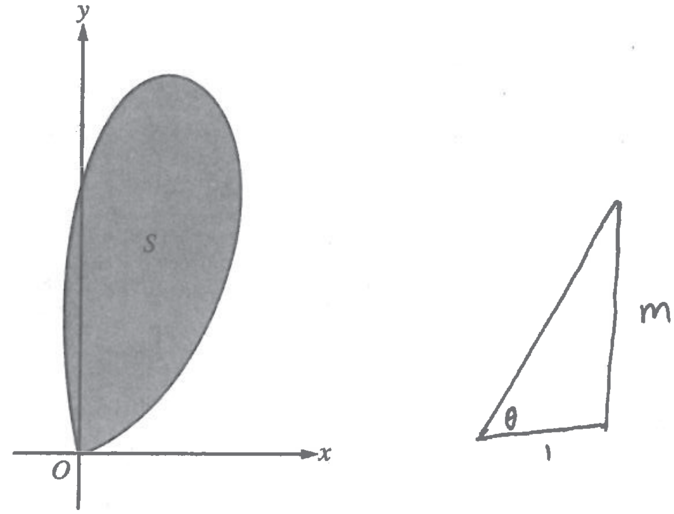
Free Response Question 2

- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 2

(a) $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292$ The area of S is 3.534.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $\frac{1}{\sqrt{\pi} - 0}\int_0^{\sqrt{\pi}} r(\theta) \, d\theta = 1.579933$ The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579).	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(c) $\tan\theta = \frac{y}{x} = m \;\Rightarrow\; \theta = \tan^{-1} m$ $\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta\right)$	$3 : \begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \quad \text{applied to } m \text{ as limit of} \\ \quad \text{integration} \\ 1 : \text{equation} \end{cases}$
(d) As $k \to \infty,$ the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis. $\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta$ $= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta = 3.324$	$2 : \begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

2A
1 of 2



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

$$A_S = \frac{1}{2}\int_0^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \boxed{3.534}$$

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

$r(\theta)$ gives the distance from the origin to a point on the curve. Therefore, the average distance on $0 \le \theta \le \sqrt{\pi}$ is

$$\frac{\int_0^{\sqrt{\pi}} 3\sqrt{\theta}\sin(\theta^2)\, d\theta}{\sqrt{\pi} - 0} = \boxed{1.580}$$

Continue question 2 on page 7.

2A
2 of 2

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

$$\frac{1}{2}\int_0^{\arctan(m)} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \frac{1}{2}\int_{\arctan(m)}^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta$$

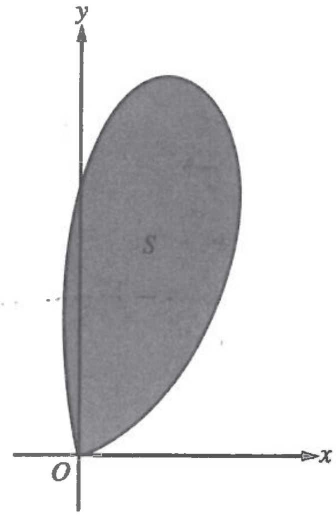
(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

$$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \boxed{3.324}$$

As k approaches ∞, the circle $r = k\cos\theta$ will cover an increasingly large portion of quadrants I and IV, but because $\cos\theta < 0$ on $(\frac{\pi}{2}, \frac{3\pi}{2})$ and $\cos\theta > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, the circle will never cover the portion of S where $\frac{\pi}{2} < \theta < \sqrt{\pi}$.

GO ON TO THE NEXT PAGE.

2B
1 of 2



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

$$\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = \boxed{3.534}$$

$$\frac{1}{2}\int_0^{\sqrt{\pi}} (3\sqrt{\theta}\sin(\theta^2))^2 d\theta$$

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

average r value

$$\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} r(\theta)\,d\theta = \boxed{1.580}$$

$$\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} (3\sqrt{\theta}\sin(\theta^2))\,d\theta$$

Do not write beyond this border.

Continue question 2 on page 7.

2B
2of2

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

$$\frac{1}{2}\int_0^m (r(\theta))\,d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}} (r(\theta))\,d\theta$$

$$\frac{1}{2}\int_m^m \left(3\sqrt{\theta}\sin(\theta^2)\right)d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)d\theta$$

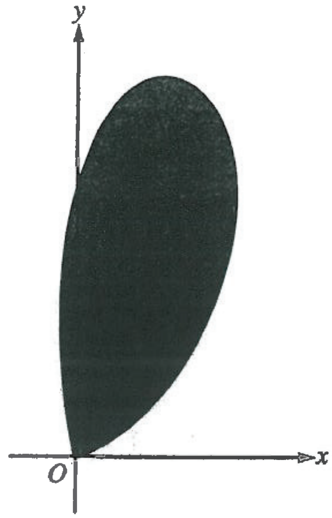
(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

$$\boxed{\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2\,d\theta = 3.324}$$

As $k\to\infty$, the circle $r=k\cos\theta$ grows to encompass greater and greater proportions of the first and fourth quadrants. As $k\to\infty$ $r=k\cos\theta$ encompasses all of S that exists in the first quadrant, which can be represented as $\frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2\,d\theta$. This is because circles represented by $r=k\cos\theta$ when $r>0$ start at the origin and expand from there.

GO ON TO THE NEXT PAGE.

2C
1 of 2



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

$$A = \frac{1}{2}\int_0^{\sqrt{\pi}} \left[3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \boxed{3.534}$$

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

$$\text{avg} = \frac{1}{\sqrt{\pi}}\int \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta =$$

Continue question 2 on page 7.

2C
2 of 2

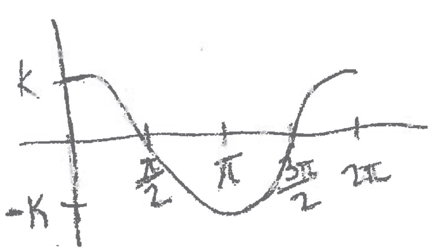
(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

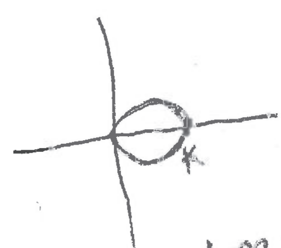
$$\frac{1}{2}\int_0^m \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta$$

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

$3\sqrt{\theta}\sin(\theta^2) = k\cos\theta$

$r = k\cos\theta$





Area = parts that overlap

$\lim_{k\to\infty} A(k) \approx$ part of S that is in quadrant 1 $\approx \boxed{3.534}$

GO ON TO THE NEXT PAGE.

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a region S is shown in an accompanying figure, and S is identified as the region enclosed by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$.
In part (a) students were asked to find the area of S. A response should demonstrate knowledge of the form of the integral that gives the area of a simple polar region and evaluate $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2\, d\theta$ using the numerical integration capability of a graphing calculator.

In part (b) students were asked for the average distance from the origin to a point on the polar curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$. A response should observe that the distance from the origin to a point on the polar curve is given simply by $r(\theta)$ and then should demonstrate that the average value of $r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is given by dividing the definite integral of $r(\theta)$ across the interval by the width of the interval. The resulting integral expression should be evaluated using the numerical integration capability of a graphing calculator.

In part (c) m denotes the positive slope of a line through the origin that divides the region S into two regions of equal areas. Students were asked to write an equation involving one or more integrals whose solution gives the value of m. A response should express the polar angle θ formed by the line and the polar axis in terms of m (namely, $\theta = \tan^{-1} m$) and use this as an upper limit in an integral that corresponds to polar area within an equation satisfying the given requirements.

In part (d) it is given that $A(k)$ represents the area of the portion of region S that is inside the circle $r = k\cos\theta$, and students were asked for the value of $\lim\limits_{k\to\infty} A(k)$. A response should observe that any point to the right of the y-axis will eventually be inside the circle $r = k\cos\theta$ for k sufficiently large. Thus $\lim\limits_{k\to\infty} A(k)$ is the area of the portion of S inside the first quadrant, computed as $\frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2\, d\theta$. The resulting integral expression should be evaluated using the numerical integration capability of a graphing calculator.

For part (a) see LO CHA-5.D/EK CHA-5.D.2, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1, LO LIM-5.A/EK LIM-5.A.3. For part (c) see LO CHA-5.D/EK CHA-5.D.1. For part (d) see LO CHA-5.D/EK CHA-5.D.2, LO LIM-5.A/EK LIM-5.A.3. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes and Practice 4: Communication and Notation.

Sample: 2A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the response earned the first point for the integral $\int_0^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta$. The factor of $\frac{1}{2}$ is not part of this point. The second point was earned for the boxed answer 3.534. In part (b) the response earned the first point for the integral $\int_0^{\sqrt{\pi}} 3\sqrt{\theta}\sin\left(\theta^2\right) d\theta$ in line 3. The denominator $\sqrt{\pi} - 0$ is not part of this point. The second point was earned for the answer 1.580 in line 3. In part (c) the response earned the first and third points for the

Question 2 (continued)

equation $\frac{1}{2}\int_0^{\arctan(m)}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta = \frac{1}{2}\int_{\arctan(m)}^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta.$ The first point is for equating polar areas; in this case, the area of the region from 0 to $\arctan(m)$ is equal to the area of the region from $\arctan(m)$ to $\sqrt{\pi}$. The third point is for a correct equation. The second point was earned for the limit $\arctan(m)$ in the definite integrals. In part (d) the response earned the first point for the limits of 0 and $\frac{\pi}{2}$ on the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$ in line 1. The second point was earned for the answer 3.324 in line 1 in the presence of the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta.$ The commentary that is in lines 2–5 is correct but not required to earn any points.

Sample: 2B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the response earned the first point for the integral $\int_0^{\sqrt{\pi}}(r(\theta))^2\,d\theta$ in line 1. The factor of $\frac{1}{2}$ is not part of this point. The second point was earned for the answer 3.534 in line 1. The definite integral $\frac{1}{2}\int_0^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$ in line 2 is a correct restatement of line 1. In part (b) the response earned the first point for the integral $\int_0^{\sqrt{\pi}} r(\theta)\,d\theta$ in line 2. The factor of $\frac{1}{\sqrt{\pi}-0}$ is not part of this point. The second point was earned for the answer 1.580 in line 2. The definite integral $\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)d\theta$ in line 3 is a correct restatement of line 2. In part (c) the response did not earn the first point because the definite integrals $\frac{1}{2}\int_0^m(r(\theta))\,d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}}(r(\theta))\,d\theta$ in line 1 and $\frac{1}{2}\int_0^m\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)d\theta$ in line 2 do not represent polar area because the expression for $r(\theta)$ is not squared. The second point was not earned because the limit of integration does not involve an inverse trigonometric function applied to m. The response is not eligible for the third point because the third point requires that both the first and second points have been earned. In part (d) the response earned the first point for the limits of 0 and $\frac{\pi}{2}$ on the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}(r(\theta))^2\,d\theta.$ The second point was earned for the answer 3.324 in line 1 in the presence of the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}(r(\theta))^2\,d\theta.$ The commentary below the boxed work is correct but not required to earn any points.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d). In part (a) the response earned the first point for the integral $\int_0^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta.$ The factor of $\frac{1}{2}$ is not

Question 2 (continued)

part of this point. The second point was earned for the correct answer 3.534. In part (b) the response did not earn the first point because an indefinite integral $\frac{1}{\sqrt{\pi}}\int\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$ is presented instead of a definite integral; additionally, $\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2$ appears as the integrand instead of $3\sqrt{\theta}\sin\left(\theta^2\right)$. The second point was not earned because no numerical answer is presented. In part (c) the response earned the first point for equating polar areas with $\frac{1}{2}\int_0^m\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$; in this case, the area of the region from 0 to m is equal to the area of the region from m to $\sqrt{\pi}$. The second point was not earned because the limit of integration does not involve an inverse trigonometric function applied to m. The response is not eligible for the third point because the third point requires that both the first and second points have been earned. In part (d) the first point was not earned because no definite integral is presented with limits 0 and $\frac{\pi}{2}$. The second point was not earned because the answer 3.534 in line 4 on the right is incorrect and is not in the presence of a definite integral.